Zygmund Type Estimates for Double Singular Cauchy-Stieltjes Integral

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Abstract:
For the double singular Cauchy-Stiltjes integral over a set of a bicylindric domain, a Zygmund type estimate connecting partial and mixed moduli of continuity of the singular integral and its density is obtained. On this basis, some spaces are constructed invariant with respect to the double singular integral.
Keywords: *Zygmund estimate, double singular integral, partial and mixed continuity modulus, invariant spaces.*

1. Introduction

Let γ^k be a closed Jordan rectifiable curve (c.j.r.c) on the complex plane z_k (k=1,2) which divides the complex plane into two parts the interior D_k^+ and the exterior D_k^-. The curves γ^1 and γ^2 define four bicylindric domains $D^\pm = D_1^\pm \times D_2^\pm$
With the boundaries having the common part $\Delta = \gamma^1 * \gamma^2$ known as spanning set. Let

$$\Phi_\psi(z) = \frac{1}{(2\pi i)^2} \int_\Delta \frac{f(s)d\psi(s)}{\prod_{k=1}^{2}(s_k - z_k)} \qquad (1)$$

be the double Cauchy-Stiltjes type integral, where $z=(z_1,z_2)$, $s=(s_1,s_2)$, $d\psi(s)=d\psi_1(s)d\psi_2(s)$, $f(s)\in C_\Delta$, where C_Δ is the space of continuous functions on $\Delta, \psi_k(s)$ being functions of bounded variation on γ^k $(k=1,2)$. Under the investigation of limiting values of the function $\Phi_\psi(z)$ there appear the following singular integrals:

$$g_\psi^{1,1}(t) = \int_\Delta \frac{\left(\overset{1,1}{\Delta} f\right)(s;t)d\psi(s)}{\prod_{k=1}^{2}(s_k - z_k)}, \quad g_\psi^{1,0}(t) = \int_{\gamma^1} \frac{\left(\overset{1,0}{\Delta} f\right)(s;t)d\psi(s_1)}{s_1 - t_1},$$

$$g_\psi^{0,1}(t) = \int_{\gamma^2} \frac{\left(\overset{0,1}{\Delta} f\right)(s;t)d\psi(s_2)}{s_2 - t_2}, \qquad (2)$$

where

$\Delta f(s;t) = f(s_1,s_2) - f(s_1,t_2) - f(t_1,s_2) + f(t_1,t_2)$,
$\Delta f(s_{t_2};t) = f(s_1,t_2) - f(t_1,t_2)$, $\Delta f(s_{t_1};t) = f(t_1,s_2) - f(t_1,t_2)$.

We denote

$$\tilde{f}_\psi(t) = g_\psi^{1,1}(t) + g_{\psi_1}^{1,0}(t) + g_{\psi_2}^{0,1}(t). \qquad (3)$$

In the case when $\psi_i(t)=t$ $(i=1,2)$, we write $\tilde{f}(t) = g^{1,1}(t) + g^{1,0}(t) + g^{0,1}(t)$.

In the case when $\gamma^i (i=1,2)$ is the unit circle the singular integral $g^{1,1}(t)$ is reduced to

$$h(x,y) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+t, y+s) ctg\frac{t}{2} ctg\frac{s}{2} dtds.$$

It is known that the space $H^2(\varphi_1,\varphi_2)$ (see [4]) is not invariant for the singular integral *h*, see [1]-[3] in the case $\varphi_1(\delta)=\varphi_2(\delta)=\delta^\alpha$, $0<\alpha<1$ and [4] or the general case. At the same time in [5] it was proved that the function spaces
$L^{\alpha,\beta} = \left\{ f \in C_{[-\pi,\pi]^2} : \omega_f(\delta_1,\delta_2) = O(\delta_1^\alpha, \delta_2^\beta), 0<\alpha,\beta<1 \right\}$
are invariant for the singular integral *h*.

In [6] for the function *h* there was obtained an estimate of Zygmund type. Based on that estimate in [7] there was constructed the class $\Im_{\infty,\infty}$ invariant with respect to the singular integral *h*. in the one-dimensional case this class was introduced in [8]. In [4] for the function *h* there was proved some analogue of the Plemeli-Privalov theorem.

In [9] there was proved an analogue of Zygmund estimate for the function $\tilde{f}(t)$, in terms of the characteristic $\theta(\delta)$ introduced in [10].

In the one-dimensional case in [11] there was obtained some of Zygmund type for the singular integral $\tilde{f}_\psi$ in terms of characteristic $\theta_\psi(\delta)$ introduced in the same paper and an analogue of Plemeli-Privalov and Magnaradze [8] theoerms were obtained.

An analogue of Zygmund estimate in terms of the characteristic $\theta(\delta)$ was obtained in [12] in the *n*-dimensional case for the function $\tilde{f}$.

In this paper we give a Zygmund type estimate connecting partial and mixed continuity moduli of the functions f and $\tilde{f}_\psi$. With the help of these estimates a Banach space invariant with respect to the singular integral $\tilde{f}_\psi$ is constructed.

2. Preliminaries

As in [11], we denote

$\theta_k^{\psi_k} = \int_{\gamma_\delta^k(t_k)} |d\psi_k(s_k)|$, $\theta_k^{\psi_k}(\delta) = \sup_{t_k\in\gamma^k} \theta_k^{\psi_k}(t_k,\delta)$,

where $\gamma_\delta^k(t_k) = \{s\in\gamma^k : |s-t_k| \le \delta,\ \delta\in(0,d_k]\}$, $d_k = \sup|s_k - t_k|$, k=1,2. Functions $\varphi_1(\delta)$ and $\varphi_2(\delta)$ non-increasing in the interval $\left[0, \int_\gamma |d\psi(s)|\right]$ are said to be equal to each other $\left(\varphi_1 \overset{ess}{=} \varphi_2\right)$, if they are equal on some dense set which contains the point $\int_\gamma |d\psi(s)|$ (see [11]).

The monotonous increasing function $\theta^\psi(\delta)$, defined by

$$\breve{\theta}^\psi(\delta) \overset{ess}{=} \sup\{t : \theta^\psi \le \delta\},\ \delta\in\left(0, \int_\gamma |d\psi(s)|\right)$$

is called the generalized inverse to the function $\theta(\delta)$, see [10,14].

Let Q be a domain in the complex plane **C** and $H(Q)$ the class of functions holomorphic in Q and continuous in $\tilde{Q}$. Let also D be a bounded region in **C** with the boundary $\partial D = \gamma$ which is (c.j.r.c). For $F\in H(C\overline{D})$ we usually assume that $\lim_{z\to\infty} F(z) = 0$.

In the case when $d\psi(t) = F(t)dt$, where $F(t)$ is limiting value of function analytic in $D^\pm$ and continuous up to the boundary, we denote

$$\theta^\psi = \theta^F(\delta) = \sup_{t\in\gamma} \theta^F(t,\delta),\ \theta^F(t,\delta) = \int_{\gamma_\delta(t)} |F(\tau)||d\tau|,\ \delta\in(0,d],$$

so that $\theta^\psi = \theta(\delta) = \sup_{t\in\gamma} \int_{\gamma_\delta(t)} |\,dt\,|$ in the case $F(t)\equiv 1$.

From the definition it follows that $\theta^F(\delta)$ it is a non-negative and non-decreasing function on (0, d] and $\theta^F(t,\delta) \le \theta^F(\delta) \le C\theta(\delta)$ with the constant C depending on F.

We denote

$$\Im_0(\gamma) = \left\{ f\in C(\gamma) : \int_0^d \frac{\omega_f(\xi)d\theta(\xi)}{\xi} < \infty \right\} \qquad (4)$$

and

$$\Im_0^F(\gamma) = \left\{ f\in C(\gamma) : \int_0^d \frac{\omega_f(\xi)d\theta^F(\xi)}{\xi} < \infty \right\} \qquad (5)$$

where $F\in H(D)$ (or $F\in H(C\overline{D})$).

Let $F\in H(D)$ (or $F\in H(C\overline{D})$). If $F(t)\ne 0\ \ \forall t\in\gamma$, then $\theta^F(\delta) \sim \theta(\delta)$ and in this case $\Im_0(\gamma) \equiv \Im_0^F(\gamma)$, that is (4)⇔(5). In the general case the conditions (4) and (5) are not equivalent. Since $\theta^F(\delta) \le C_F\theta(\delta)$, we have

$$\int_0^d \frac{\omega_f(\xi)}{\xi} d\theta^F(\xi) \le C_F \int_0^d \frac{\omega_f(\xi)}{\xi} d\theta(\xi),$$ whence $\Im_0(\gamma) \subset \Im_0^F(\gamma)$.

To study the properties of the integral (2), we arrive at the necessity to choose the following basic characteristics of functions $f\in C_\Delta$:

1) mixed modulus of continuity ($\delta = (\delta_1,\delta_2)$, $\delta_1 > 0$, $\delta_2 > 0$, $\xi = (\xi_1,\xi_2)$):

$$\overset{1,1}{\omega}_f(\delta) = \delta_1\cdot\delta_2 \sup_{\xi_1\ge\delta_1,\xi_2\ge\delta_2} \frac{\overset{1,1}{\omega}(f:\xi_1,\xi_2)}{\xi_1\xi_2} = \delta \sup_{\xi\ge\delta} \frac{\overset{1,1}{\omega}(f:\xi)}{\xi},$$

where $\overset{1,1}{\omega}(f,\delta) = \sup_{\substack{|s_1-t_1|<\delta_1\\ |s_2-t_2|<\delta_2}} \left| \left(\overset{1,1}{\Delta} f\right)(s;t) \right|$;

2) partial continuity modulus $\overset{1,0}{\omega}_f(\delta_1) = \delta_1 \sup_{\xi_1\ge\delta_1} \frac{\overset{1,0}{\omega}(f,\xi_1)}{\xi_1}$,

$\overset{1,0}{\omega}(f;\delta_1) = \sup_{t_2\in\gamma^2} \sup_{|s_1-t_1|\le\delta_1} |\Delta f(s_{t_2};t)|$ and

$\overset{0,1}{\omega}_f(\delta_2) = \delta_2 \sup_{\xi_2\ge\delta_2} \frac{\overset{0,1}{\omega}(f,\xi_2)}{\xi_2}$, $\overset{0,1}{\omega}(f;\delta_2) = \sup_{t_1\in\gamma^1} \sup_{|s_2-t_2|\le\delta_2} |\Delta f(s_{t_1};t)|$.

By $\Phi^1_{(0,d]}$ we denote the set of those non-negative increasing functions $\varphi(\delta)$ on (0, d] for which $\lim_{\delta\to 0} \varphi(\delta) = 0$ and $\delta^{-1}\varphi(\delta)$ decreases.

Let $\Phi_{(0,d_1]\times(0,d_2]} = \Phi_{T^2}$ denote the set of functions $\omega(\delta_1,\delta_2) = \omega(\delta)$ defined on $T^2 = (0,d_1]\times(0,d_2]$ and belonging to Φ^1 in each argument, i.e.

1) $\omega(\delta)\in\Phi^1_{(0,d_2]}$ in δ_2 for any fixed δ_1;

2) $\omega(\delta)\in\Phi^1_{(0,d_1]}$ in δ_1 for any fixed δ_2.

In [15] it was shown that the properties 1) and 2) are characteristic for continuity modulus in the sense that for every $\omega\in\Phi_{T^2}$ there exist such a function $f\in C_\Delta$

$\omega_f(\delta_1,\delta_2) \sim \omega(\delta_1,\delta_2)$, $\overset{1,0}{\omega}_f(\delta_1) \sim \omega(\delta_1,d_2)$, $\overset{0,1}{\omega}_f(\delta_2) \sim \omega(d_1,\delta_2)$

By V_γ (see [11]) we denote the set of all functions with a bounded variation on γ for which the integral

$$\left| \int_{\gamma\setminus\gamma_\varepsilon(t)} \frac{d\psi(s)}{\prod_{k=1}^2 (s_k - t_k)} \right|$$ is uniformly bounded.

Let function $\omega(\delta_1,\delta_2)$ be defined on T^2 non-negative and satisfying the condition

$$\int_0^{d_1}\int_0^{d_2} \frac{\omega(\xi_1,\xi_2)}{\xi_1\xi_2} d\theta_1^{\psi_1}(\xi_1) d\theta_2^{\psi_2}(\xi_2) < \infty .$$

We introduce the Zygmund type operator

$Z(\omega;\delta,\theta^\psi,\breve{\theta}^\psi) = Z(\omega;\delta_1,\delta_2,\theta_1^{\psi_1},\theta_2^{\psi_2},\breve{\theta}_1^{\psi_1},\breve{\theta}_2^{\psi_2}) =$

$$=\int_0^{\theta_1^{\psi_1}(\delta_1)}\int_0^{\theta_2^{\psi_2}(\delta_2)}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2+$$

$$+\delta_1\int_{\theta_1^{\psi_1}(\delta_1)}^{d_1}\int_0^{\theta_2^{\psi_2}(\delta_2)}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\left[\breve{\theta}_1^{\psi_1}(\xi_1)\right]^2\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2+$$

$$+\delta_2\int_0^{\theta_1^{\psi_1}(\delta_1)}\int_{\theta_2^{\psi_2}(\delta_2)}^{d_2}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot\left[\breve{\theta}_2^{\psi_2}(\xi_2)\right]^2}d\xi_1 d\xi_2+$$

$$+\delta_1\delta_2\int_{\theta_1^{\psi_1}(\delta_1)}^{d_1}\int_{\theta_2^{\psi_2}(\delta_2)}^{d_2}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\left[\breve{\theta}_1^{\psi_1}(\xi_1)\right]^2\cdot\left[\breve{\theta}_2^{\psi_2}(\xi_2)\right]^2}d\xi_1 d\xi_2 .$$

It isn't hard to show that $Z\left(\omega;\delta,\theta^{\psi},\breve{\theta}^{\psi}\right)\in\Phi_{T^2}$ and

$$Z\left(\omega;\delta,\theta^{\psi},\breve{\theta}^{\psi}\right)=Z_1\left(Z_2\left(\omega\left(\breve{\theta}_1^{\xi_1},\cdot\right),\delta_2,\theta_2{}^{\psi_2},\breve{\theta}_2{}^{\psi_2}\right),\delta_1,\theta_1{}^{\psi_1},\breve{\theta}_1{}^{\psi_1}\right)=$$

$$=\int_0^{\theta_1^{\psi_1}(\delta_1)}\frac{Z_2\left(\omega\left(\breve{\theta}_1^{\xi_1},\cdot\right),\delta_2,\theta_2{}^{\psi_2},\breve{\theta}_2{}^{\psi_2}\right)}{\breve{\theta}_1^{\psi_1}(\xi_1)}d\xi_1+$$

$$+\delta_1\int_{\theta_1^{\psi_1}(\delta_1)}^{d_1}\frac{Z_2\left(\omega\left(\breve{\theta}_1^{\xi_1},\cdot\right),\delta_2,\theta_2{}^{\psi_2},\breve{\theta}_2{}^{\psi_2}\right)}{\left[\breve{\theta}_1^{\psi_1}(\xi_1)\right]^2}d\xi_1$$

and

$$Z_2\left(\omega\left(\breve{\theta}_1^{\xi_1},\cdot\right),\delta_2,\theta_2{}^{\psi_2},\breve{\theta}_2{}^{\psi_2}\right)=$$

$$=\int_0^{\theta_2^{\psi_2}(\delta_2)}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_2+$$

$$+\delta_2\int_{\theta_2^{\psi_2}(\delta_2)}^{d_2}\frac{\omega\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\left[\breve{\theta}_2^{\psi_2}(\xi_2)\right]^2}d\xi_2$$

For further goals we need the following technical lemma (see [11]).

Lemma 1. Let $g(t)$ be a function non-decreasing on $(0,d]$ and $\psi(t)$ a function of bounded variation on γ then $\forall\varepsilon',\varepsilon''\in(0,d]$, $\varepsilon'<\varepsilon''$

$$\int_{\gamma_{\varepsilon''}(t)\backslash\gamma_{\varepsilon'}(t)}g(|s-t|)|d\psi(s)|=\int_{\varepsilon'}^{\varepsilon''}g(s)d\theta^{\psi}(t,s).$$

Let us denote

$$\overset{1,1}{g}_{\psi,\varepsilon}(t)=\int_{\Delta\backslash\Delta_\varepsilon}\frac{\Delta f(s;t)}{s-t}d\psi(s),$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2)$, so that $\Delta\backslash\Delta_\varepsilon=\gamma^1\backslash\gamma^1_{\varepsilon_1}(t_1)\times\gamma^2\backslash\gamma^2_{\varepsilon_2}(t_2)$.

As usual we say that the integral $\overset{1,1}{g}_{\psi}$ exists in the principal value sense, if there exists the finite limit

$$\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\overset{1,1}{g}_{\psi,\varepsilon_1,\varepsilon_2}.$$

Theorem 1. Let $f\in C_\Delta,\psi_k\in V_\gamma\,(k=1,2)$. If

$$\int_0^d\frac{\overset{1,1}{\omega}_f(\eta)}{\eta}d\theta^{\psi}(\eta)=\int_0^{d_1}\int_0^{d_2}\frac{\overset{1,1}{\omega}_f(\eta_1,\eta_2)}{\eta_1\eta_2}d\theta_1{}^{\psi_1}(\eta_1)d\theta_2{}^{\psi_2}(\eta_2)<\infty,$$

then the limits $\lim_{\varepsilon_1\to 0}\overset{1,1}{g}_{\psi,\varepsilon}(t_1,t_2)$, $\lim_{\varepsilon_2\to 0}\overset{1,1}{g}_{\psi,\varepsilon}(t_1,t_2)$, $\lim_{\substack{\varepsilon_1\to 0\\ \varepsilon_2\to 0}}\overset{1,1}{g}_{\psi,\varepsilon}(t_1,t_2)$ with any fixed $\varepsilon_2\in(0,d_2]$ in the first limit and any fixed $\varepsilon_1\in(0,d_1]$ in the second one exist uniformly in t_1,t_2.

Proof. It can be seen directly that

$$\overset{1,1}{g}_{\varepsilon_1,\varepsilon_2}(t_1,t_2)-\overset{1,1}{g}_{\eta_1,\eta_2}(t_1,t_2)=$$

$$=\int_{\gamma^1_{\eta_1}(t_1)\backslash\gamma^1_{\varepsilon_1}(t_1)}\int_{\gamma^2_{\eta_2}(t_2)\backslash\gamma^2_{\varepsilon_2}(t_2)}\frac{\Delta f(s;t)}{s-t}d\psi(s)+$$

$$+\int_{\gamma^1\backslash\gamma^1_{\varepsilon_1}(t_1)}\int_{\gamma^2_{\eta_2}(t_2)\backslash\gamma^2_{\varepsilon_2}(t_2)}\frac{\Delta f(s;t)}{s-t}d\psi(s)+$$

$$+\int_{\gamma^1_{\eta_1}(t_1)\backslash\gamma^1_{\varepsilon_1}(t_1)}\int_{\gamma^2(t_2)\backslash\gamma^2_{\varepsilon_2}(t_2)}\frac{\Delta f(s;t)}{s-t}d\psi(s)=$$

$$=\Im_1+\Im_2+\Im_3,$$

where $0<\varepsilon_1<\eta_1\le d_1$ and $0<\varepsilon_2<\eta_2\le d_2$.

We estimate every term separately. Obviously

$$|\Im_1|\le\int_{\gamma^1_{\eta_1}(t_1)\backslash\gamma^1_{\varepsilon_1}(t_1)}\int_{\gamma^2_{\eta_2}(t_2)\backslash\gamma^2_{\varepsilon_2}(t_2)}\frac{\overset{1,1}{\omega}_f(|s_1-t_1|;|s_2-t_2|)}{|s_1-t_1|\cdot|s_2-t_2|}\times$$

$$\times|d\psi_1(s_1)|\cdot|d\psi_2(s_2)|.$$

Case 1) $\eta_1\le\varepsilon_1$. Applying Lemma 1 and Theorem subsequently from [14] and taking into account that $\theta_k^{\psi_k}(t_k,\delta)\le\theta_k^{\psi_k}(\delta)$ and $\dfrac{\overset{1,1}{\omega}_f(\delta_1,\delta_2)}{\delta_k}\downarrow(k=1,2)$, we obtain

$$|\Im_1|\le\int_0^{\eta_1}\int_0^{\eta_2}\frac{\overset{1,1}{\omega}_f(\xi_1,\xi_2)}{\xi_1\xi_2}d\theta_1^{\psi_1}(\xi_1)d\theta_2^{\psi_2}(\xi_2).$$

Case 2) $\varepsilon_1\le\varepsilon_2\le\eta_1\le\eta_2$. In this case we have

$$|\Im_1|\le\left(\int_{\gamma^2_{\varepsilon_2}(t_2)\backslash\gamma^1_{\varepsilon_1}(t_1)}+\int_{\gamma^1_{\eta_1}(t_1)\backslash\gamma^2_{\varepsilon_2}(t_2)}\right)\left(\int_{\gamma^2_{\eta_2}(t_2)\backslash\gamma^2_{\varepsilon_2}(t_2)}+\int_{\gamma^2_{\eta_2}(t_2)\backslash\gamma^1_{\eta_1}(t_1)}\right)$$

$$\times\frac{\overset{1,1}{\omega}_f(|\xi_1-t_1|,|\xi_2-t_2|)}{|\xi_1-t_1|\cdot|\xi_2-t_2|}|d\psi_1(\xi_1)|\cdot|d\psi_2(\xi_2)|.$$

Taking into account the case 1) we obtain

$$|\Im_1|\le\left(\int_0^{\varepsilon_1}\int_0^{\varepsilon_2}+\int_0^{\varepsilon_2}\int_0^{\eta_2}+\int_0^{\varepsilon_1}\int_0^{\eta_1}+\int_0^{\eta_1}\int_0^{\eta_2}\right)\times$$

$$\times\frac{\overset{1,1}{\omega}_f(\xi_1,\xi_2)}{\xi_1\xi_2}d\theta_1^{\psi_1}(\xi_1)d\theta_2^{\psi_2}(\xi_2).$$

The integral $\Im_1$ in the cases $\varepsilon_1\le\varepsilon_2\le\eta_2\le\eta_1$, $\varepsilon_2\le\varepsilon_1\le\eta_2\le\eta_1$ $\varepsilon_2\le\varepsilon_1\le\eta_1\le\eta_2$ is estimated in a similar way.

Passing to the integral $\Im_3$, we have

$$|\Im_3|\le\int_0^{\eta_1}\int_0^{d_2}\frac{\overset{1,1}{\omega}_f(\xi_1,\xi_2)}{\xi_1\xi_2}d\theta_1^{\psi_1}(\xi_1)d\theta_2^{\psi_2}(\xi_2),$$

if $\eta_1 \le \eta_2$ and

$$|\Im_3| \le \left(\int_0^{\eta_2}\int_0^{d_2} + \int_0^{\eta_1}\int_0^{d_2} \right) \frac{\overset{1,1}{\omega}_f(\xi_1,\xi_2)}{\xi_1\xi_2} d\theta_1^{\psi_1}(\xi_1) d\theta_2^{\psi_2}(\xi_2),$$

if $\varepsilon_1 \le \eta_2 \le \eta_1$. The integral $\Im_3$ is estimated similar in the case $\eta_2 \le \varepsilon_1 \le \eta_1$. In a symmetrical way the integral $\Im_2$ is estimates prove the theorem.

In the following theorem we use the notation

$$\Im_0^{\psi}(\Delta) = \left\{ f \in C_\Delta : \int_0^{d_1}\int_0^{d_2} \frac{\overset{1,1}{\omega}_f(\xi_1,\xi_2)}{\xi_1\xi_2} d\theta_1^{\psi_1}(\xi_1) d\theta_2^{\psi_2}(\xi_2) < \infty, \right.$$

$$\left. \int_0^{d_1} \frac{\overset{1,0}{\omega}_f(\xi_1)}{\xi_1} d\theta_1^{\psi_1}(\xi_1) < \infty, \int_0^{d_2} \frac{\overset{0,1}{\omega}_f(\xi_2)}{\xi_2} d\theta_2^{\psi_2}(\xi_2) < \infty \right\}.$$

3. Main result

Teorem 2. Let $f \in \Im_0^{\psi}(\Delta)$ with $\psi = (\psi_1, \psi_2)$, $\psi_k \in V_k$, k=1,2. Then the following inequalities are valid

$$\overset{1,1}{\omega}_{\bar f}(\delta_1,\delta_2) \le C_1 Z\left(\overset{1,1}{\omega}_f; \delta; \theta^{\psi}; \breve{\theta}^{\psi}\right), \ 0 < \delta_k \le d_k, \ k=1,2$$

$$\overset{1,0}{\omega}_{\bar f}(\delta_1) \le C_2\left[Z\left(\overset{1,1}{\omega}_f; \delta_1, d_2, \theta^{\psi}, \breve{\theta}^{\psi}\right) + Z\left(\overset{1,0}{\omega}_f; \delta_1, \theta_1^{\psi_1}, \breve{\theta}_1^{\psi_1}\right)\right],$$

$$0 < \delta_1 \le d_1, \qquad (6)$$

$$\overset{0,1}{\omega}_{\bar f}(\delta_2) \le C_3\left[Z\left(\overset{1,1}{\omega}_f; d_1, \delta_2, \theta^{\psi}, \breve{\theta}^{\psi}\right) + Z\left(\overset{0,1}{\omega}_f; \delta_2, \theta_2^{\psi_2}, \breve{\theta}_2^{\psi_2}\right)\right],$$

$$0 < \delta_2 \le d_2.$$

Proof. Let $t_k, t_k' \in \gamma^k$, $|t_k - t_k'| = \varepsilon_k \le d_k$, k=1,2. For the function $\bar f_\psi(t)$ we estimate mixed and partial increments

$$\Delta \bar f_\psi(t;t') = \Delta \overset{1,1}{g}_\psi(t;t') + \Delta \overset{1,0}{g}_{\psi_1}(t;t') + \Delta \overset{0,1}{g}_{\psi_2}(t;t'), \qquad (7)$$

$$\Delta \bar f_\psi(t_{t_1'};t') = \Delta \overset{1,1}{g}_\psi(t_{t_1'};t') + \Delta \overset{1,0}{g}_{\psi_1}(t_{t_1'};t') + \Delta \overset{0,1}{g}_{\psi_2}(t_{t_1'};t') \qquad (8)$$

and

$$\Delta \bar f_\psi(t_{t_2'};t') = \Delta \overset{1,1}{g}_\psi(t_{t_2'};t') + \Delta \overset{1,0}{g}_{\psi_1}(t_{t_2'};t') + \Delta \overset{0,1}{g}_{\psi_2}(t_{t_2'};t'). \qquad (9)$$

To estimate $\Delta \overset{1,1}{g}_\psi(t;t')$ we observe that

$$\Delta \overset{1,1}{g}_\psi(t;t') = \int_{\gamma^1} \frac{g_2(\Delta f; s_{t_2}, t_{t_2'})}{s_1 - t_1} d\psi_1(s_1) - \int_{\gamma^1} \frac{g_2(\Delta f; s_{t_2}, t')}{s_1 - t_1'} d\psi_1(s_1) \qquad (10)$$

where

$$g_2(\Delta f; s_{t_2}, t_{t_2'}) = \int_{\gamma^2} \frac{\Delta f(s;t)}{s_2 - t_2} d\psi_2(s_2) - \int_{\gamma^2} \frac{\Delta f(s;t_{t_2'})}{s_2 - t_2'} d\psi_2(s_2) \qquad (11)$$

and similarly for $g_2(\Delta f; s_{t_2}, t')$.

It's easy to see that $\Delta f(s;t) - \Delta f(s;t_{t_k'}) = -\Delta f(s_{t_k}; t_{t_k'}), k=1,2$ so that

$$g_2(\Delta f; s_{t_2}, t_{t_2'}) =$$

$$= \int_{\gamma_{\varepsilon_2}^2(t_2)} \frac{\Delta f(s;t)}{s_2 - t_2} d\psi_2(s_2) - \int_{\gamma_{\varepsilon_2}^2(t_2')} \frac{\Delta f(s;t)}{s_2 - t_2} d\psi_2(s_2) +$$

$$+ (t_2 - t_2') \int_{\gamma^2 \setminus \gamma_{\varepsilon_2}^2} \frac{\Delta f(s;t)}{(s_2 - t_2)(s_2 - t_2')} d\psi_2(s_2) -$$

$$- \int_{\gamma^2 \setminus \gamma_{\varepsilon_2}^2} \frac{\Delta f(s_{t_2}; t_{t_2'})}{s_2 - t_2'} d\psi_2(s_2) + \int_{\gamma_{\varepsilon_2}^2(t_2') \setminus \gamma_{\varepsilon_2}^2(t_2)} \frac{\Delta f(s;t)}{s_2 - t_2} d\psi_2(s_2) +$$

$$+ \int_{\gamma_{\varepsilon_2}^2(t_2) \setminus \gamma_{\varepsilon_2}^2(t_2')} \frac{\Delta f(s;t_{t_2'})}{s_2 - t_2'} d\psi_2(s_2) = \sum_{k=0}^{6} \Im_k, \qquad (12)$$

where $\gamma_{\varepsilon_k}^k = \gamma_{\varepsilon_k}^k(t_k) \cup \gamma_{\varepsilon_k}^k(t_k')$ k=1,2. Basing on the of the proof of Theorem 1 from [11] and Lemma 1, it's easy to obtain the estimates

$$|\Im_i| \le \int_0^{\varepsilon_2} \frac{\overset{1,1}{\omega}_f(|s_1 - t_1|, \xi_2)}{\xi_2} d\theta_2^{\psi_2}(\xi_2), \ i=1,2,$$

$$|\Im_3| \le \varepsilon_2 \int_{\varepsilon_2}^{d_2} \frac{\overset{1,1}{\omega}_f(|s_1 - t_1|, \xi_2)}{\xi_2^2} d\theta_2^{\psi_2}(\xi_2),$$

$$|\Im_4| \le 3\varepsilon_2 \int_{\varepsilon_2}^{d_2} \frac{\overset{1,1}{\omega}_f(|s_1 - t_1|, \xi_2)}{\xi_2^2} d\theta_2^{\psi_2}(\xi_2) + M \overset{1,1}{\omega}_f(|s_1 - t_1|, \varepsilon_2),$$

where $M = \sup \left| \int_{\gamma^2 \setminus \gamma_{\varepsilon_2}^2} \frac{d\psi_2(s_2)}{s_2 - t_2'} \right|$ and

$$|\Im_i| \le 2\varepsilon_2 \int_{\varepsilon_2}^{d_2} \frac{\overset{1,1}{\omega}_f(|s_1 - t_1|, \xi_2)}{\xi_2^2} d\theta_2^{\psi_2}(\xi_2), \ i=5,6.$$

Gathering the estimates for the integrals $\Im_i$ (i=1,2,…,6) and taking into account the theorem from [14], p.241, we obtain

$$|g_2(\Delta f; s_{t_2}; t_{t_2'})| \le C\left[\int_0^{\theta_{t_2}^{\psi_2}(\varepsilon_2)} \frac{\overset{1,1}{\omega}_f(|s_1 - t_1|, \breve{\theta}_{t_2}^{\psi_2}(\xi_2))}{\breve{\theta}_{t_2}^{\psi_2}(\xi_2)} d\xi_2 + \right.$$

$$+ \varepsilon \int_{\theta_{t_2}^{\psi_2}(\varepsilon_2)}^{d_2} \frac{\overset{1,1}{\omega}_f(|s_1 - t_1|, \breve{\theta}_{t_2}^{\psi_2}(\xi_2))}{[\breve{\theta}_{t_2}^{\psi_2}(\xi_2)]^2} d\xi_2 +$$

$$\left. + \overset{1,1}{\omega}_f(|s_1 - t_1, \xi_2|) \right]. \qquad (13)$$

Since the function

$$F(\delta,\eta) = \int_0^{\eta} \frac{\overset{1,1}{\omega}_f(\delta, \breve{\theta}_{t_2}^{\psi_2}(\xi_2))}{\breve{\theta}_{t_2}^{\psi_2}(\xi_2)} d\xi_2 + \eta \int_{\eta}^{d_2} \frac{\overset{1,1}{\omega}_f(\delta, \breve{\theta}_{t_2}^{\psi_2}(\xi_2))}{[\breve{\theta}_{t_2}^{\psi_2}(\xi_2)]^2} d\xi_2$$

is increasing in η for any $\delta \in (0, d_1]$ and $\breve{\theta}_2^{\psi_2}(\xi_2) \le \breve{\theta}_{t_2}^{\psi_2}(\xi_2)$, we obtain

$$\overset{1,1}{\omega}_f\left(|s_1-t_1|,\varepsilon_2\right)\le\int_0^{\theta_2^{\psi_2}(\varepsilon_2)}\frac{\overset{1,1}{\omega}_f\left(|s_1-t_1|,\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_2 .$$

From the inequality (13) taking $\varepsilon_2\le\delta$ we have

$$\left|g_2\left(\Delta f;s_{t_2},t_{t_2'}\right)\right|\le C\left(\int_0^{\theta_2^{\psi_2}(\delta_2)}\frac{\overset{1,1}{\omega}_f\left(|s_1-t_1|,\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_2+\right.$$

$$\left.+\delta_2\int_{\theta_2^{\psi_2}(\delta_2)}^{d_2}\frac{\overset{1,1}{\omega}_f\left(|s_1-t_1|,\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\left[\breve{\theta}_2^{\psi_2}(\xi_2)\right]^2}d\xi_2\right)=$$

$$=Z_2\left(\overset{1,1}{\omega}_f\left(|s_1-t_1|,\delta_2,\theta_2^{\psi_2},\breve{\theta}_2^{\psi_2}\right)\right). \qquad (14)$$

Here and below, c denote absolute constants, which are different at different in different inequalities.

The equality (10) differs from (11) by the fact that in (10) the integration is taken over through γ^1 no γ^2 and the density Δf is replaced by g_2. Therefore to derive the estimate for the difference (10) we may use the partition of this difference similar to that in (12). We have

$$\Delta\overset{1,1}{g}_\psi(t,t')=\int_{\gamma^1_{\varepsilon_1}(t_1)}\frac{g_2\left(\Delta f;s_{t_2},t_{t_2'}\right)}{s_1-t_1}d\psi_1(s_1)-$$

$$-\int_{\gamma^1_{\varepsilon_1}(t_1)}\frac{g_2\left(\Delta f;s_{t_2},t'\right)}{s_1-t_1}d\psi_1(s_1)+$$

$$+\left(t_1-t_1'\right)\int_{\gamma^1\setminus\gamma^1_{\varepsilon_1}}\frac{g_2\left(\Delta f;s_{t_2},t_{t_2'}\right)}{\left(s_1-t_1\right)\left(s_1-t_1'\right)}d\psi_1(s_1)+$$

$$+\int_{\gamma^1_{\varepsilon_1}(t_1')\setminus\gamma^1_{\varepsilon_1}(t_1)}\frac{g_2\left(\Delta f;s_{t_2},t_{t_2'}\right)}{s_1-t_1}d\psi_1(s_1)-$$

$$-\int_{\gamma^1_{\varepsilon_1}(t_1')\setminus\gamma^1_{\varepsilon_1}(t_1)}\frac{g_2\left(\Delta f;s_{t_2},t'\right)}{s_1-t_1'}d\psi_1(s_1)+$$

$$+\int_{\gamma\setminus\gamma^1_{\varepsilon_1}}\frac{g_2\left(\Delta f;s_{t_2},t_{t_2'}\right)-g_2\left(\Delta f;s_{t_2},t'\right)}{s_1-t_1'}d\psi_1(s_1), \qquad (15)$$

with $g_2\left(\Delta f;s_{t_2},t_{t_2'}\right)-g_2\left(\Delta f;s_{t_2},t'\right)=$

$$=\int_{\gamma^2}\frac{\Delta f\left(s_{t_1'};t\right)}{s_2-t_2}d\psi_2(s_2)-\int_{\gamma^2}\frac{\Delta f\left(s_{t_1'};t_{t_2'}\right)}{s_2-t_2'}d\psi_2(s_2). \qquad (16)$$

We denote the terms in the right-hand side of (15) as i_k, $k=\overline{1,6}$.

Taking into account the estimate for the difference (11) we can obtain the following estimate for (16)

$\left|g_2\left(\Delta f;s_{t_2},t_{t_2'}\right)-g_2\left(\Delta f;s_{t_2},t'\right)\right|\le$

$$\le C\left(\int_0^{\theta^{\psi_2}(\varepsilon_2)}\frac{\overset{1,1}{\omega}_f\left(|s_1-t_1|,\breve{\theta}^{\psi_2}(\varepsilon_2)\right)}{\breve{\theta}^{\psi_2}(\varepsilon_2)}d\xi_2^{\psi_2}(\xi_2)+\right.$$

$$\left.+\varepsilon_2\int_{\theta^{\psi_2}(\varepsilon_2)}^{d_2}\frac{\overset{1,1}{\omega}_f\left(|s_1-t_1|,\breve{\theta}^{\psi_2}(\varepsilon_2)\right)}{\left[\breve{\theta}^{\psi_2}(\varepsilon_2)\right]^2}d\xi_2^{\psi_2}(\xi_2)\right)=$$

$$=Z_2\left(\overset{1,1}{\omega}_f\left(|s_1-t_1|\right),\varepsilon_2,\theta^{\psi_2},\breve{\theta}^{\psi_2}\right). \qquad (17)$$

Keeping the bounds (14) and (17) in mind we may estimate the integrals i_k, $k=\overline{1,6}$ in the same way as it was done for the integrals $\Im_k$, $k=\overline{1,6}$ and then for $\varepsilon_1\le\delta_1$ we get

$$\left|\Delta\overset{1,1}{g}_\psi(t;t')\right|\le C_1\left(\int_0^{\theta_1^{\psi_1}(\delta_1)}\frac{Z_2\left(\overset{1,1}{\omega}_f\left(\breve{\theta}_1(\xi_1),\cdot\right),\delta_2,\theta_2^{\psi_2},\breve{\theta}_2^{\psi_2}\right)}{\breve{\theta}_1^{\psi_1}(\xi_1)}d\xi_1+\right.$$

$$\left.+\delta_1\int_{\theta_1^{\psi_1}(\delta_1)}^{l_1}\frac{Z_2\left(\overset{1,1}{\omega}_f\left(\breve{\theta}_1(\xi_1),\cdot\right),\delta_2,\theta_2^{\psi_2},\breve{\theta}_2^{\psi_2}\right)}{\left[\breve{\theta}_1^{\psi_1}(\xi_1)\right]^2}d\xi_1\right)=$$

$$=Z_1\left[Z_2\left(\overset{1,1}{\omega}_f\left(\breve{\theta}_1(\xi_1),\cdot\right),\delta_2,\theta_2^{\psi_2},\breve{\theta}_2^{\psi_2}\right),\delta_1,\theta_1^{\psi_1},\breve{\theta}_1^{\psi_1}\right]=$$

$$=Z\left(\overset{1,1}{\omega}_f;\delta,\theta^{\psi},\breve{\theta}^{\psi}\right). \qquad (18)$$

Now we have to estimate $\Delta\overset{1,0}{g}_{\psi_1}(t;t')$ and $\Delta\overset{0,1}{g}_{\psi_2}(t;t')$.

Estimating $\Delta\overset{1,0}{g}_{\psi_1}(t;t')$ and $\Delta\overset{0,1}{g}_{\psi_2}(t;t')$ similarly to account in (11) we obtain

$$\left|\Delta\overset{1,0}{g}_{\psi_1}(t;t')\right|\le C\left(\int_0^{\theta_2^{\psi_2}(\delta_2)}\int_0^{\theta_1^{\psi_1}(\delta_1)}\frac{\overset{1,1}{\omega}_f\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1d\xi_2+\right.$$

$$\left.+\delta_1\int_0^{\theta_2^{\psi_2}(\delta_2)}\int_{\theta_1^{\psi_1}(\delta_1)}^{l_1}\frac{\overset{1,1}{\omega}_f\left(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2)\right)}{\left[\breve{\theta}_1^{\psi_1}(\xi_1)\right]^2\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1d\xi_2\right)=$$

$$=\int_0^{\breve{\theta}_2^{\psi_2}(\delta_2)}\frac{Z_1\left(\overset{1,1}{\omega}_f\left(\cdot,\breve{\theta}_2^{\psi_2}(\xi_2)\right),\delta_1,\theta_1^{\psi_1},\breve{\theta}_1^{\psi_1}\right)}{\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_2 . \qquad (19)$$

In the same way we get

$$\left|\Delta\overset{0,1}{g}_{\psi_2}(t;t')\right|\le\int_0^{\breve{\theta}_1^{\psi_1}(\delta_1)}\frac{Z_2\left(\overset{1,1}{\omega}_f\left(\breve{\theta}_1^{\psi_1}(\xi_1),\bullet\right),\delta_2,\theta_2^{\psi_2},\breve{\theta}_2^{\psi_2}\right)}{\breve{\theta}_1^{\psi_1}(\xi_1)}d\xi_1 .$$

(20)

Gathering the estimates (18), (19), (20) we finally obtain

$$\overset{1,1}{\omega}_f(\delta_1,\delta_2)=O\left(Z\left(\overset{1,1}{\omega}_f;\delta,\theta^{\psi},\breve{\theta}^{\psi}\right)\right).$$

(21)

Let us estimate $\Delta\bar{f}_\psi\left(t_{t_2'};t'\right)=\bar{f}_\psi\left(t_1,t_2'\right)-\bar{f}_\psi\left(t_1',t_2'\right)$. We have

$$\Delta\bar{f}_\psi\left(t_{t_2'};t'\right)=\int_{\gamma^2}\frac{d\psi_2(s_2)}{s_2-t_2'}\left(\int_{\gamma^1}\frac{\Delta f\left(s;t_{t_2'}\right)}{s_1-t_1}d\psi_1(s_1)-\right.$$

$$-\int_{\gamma^1}\frac{\Delta f(s;t')}{s_1-t_1'}d\psi_1(s_1)\Bigg)+\int_{\gamma^2}\frac{\Delta f\left(s_{t_1};t_{t_2'}\right)}{s_2-t_2'}d\psi_2(s_2)-$$

$$-\int_{\gamma^2}\frac{\Delta f\left(s_{t_1'};t'\right)}{s_2-t_2'}d\psi_2(s_2)+\int_{\gamma^1}\frac{\Delta f\left(s_{t_2};t_{t_2'}\right)}{s_1-t_1}d\psi_1(s_1)-$$

$$-\int_{\gamma^1}\frac{\Delta f\left(s_{t_2'};t'\right)}{s_1-t_1'}d\psi_1(s_1)=$$

$$=\Delta\overset{1,1}{g}_{\psi}\left(t_{t_2'};t'\right)+\Delta\overset{0,1}{g}_{\psi_2}\left(t_{t_2'};t'\right)+\Delta\overset{1,0}{g}_{\psi_1}\left(t_{t_2'};t'\right).$$

Now we have to estimate $\left|\Delta\overset{1,1}{g}_{\psi}\left(t_{t_2'};t'\right)\right|$. We have

$$\left|\Delta\overset{1,1}{g}_{\psi}\left(t_{t_2'};t'\right)\right|\le C\Bigg(\int_0^{\theta_1^{\psi_1}(\varepsilon_1)}\int_0^{l_2}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1),\breve\theta_2^{\psi_2}(\xi_2)\right)}{\breve\theta_1^{\psi_1}(\xi_1)\cdot\breve\theta_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2+$$

$$+\varepsilon_1\int_{\theta_1^{\psi_1}(\varepsilon_1)}^{l_1}\int_0^{l_2}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1),\breve\theta_2^{\psi_2}(\xi_2)\right)}{\left[\breve\theta_1^{\psi_1}(\xi_1)\right]^2\cdot\breve\theta_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2\Bigg).$$

To have bounds for $\Delta\overset{0,1}{g}_{\psi_2}\left(t_{t_2'};t'\right)$ and $\Delta\overset{1,0}{g}_{\psi_1}\left(t_{t_2'};t'\right)$ we use Lemma 1 and apply the theorem from [14]. Taking into account the inequality

$$\overset{1,1}{\omega}_f\left(|\varepsilon_1-t_1|,\xi_2\right)\le\int_0^{\theta_2^{\psi_2}(\varepsilon_2)}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1),\xi_2\right)}{\breve\theta_1^{\psi_1}(\xi_1)}d\xi_1,$$

We arrive at the estimate

$$\left|\Delta\overset{0,1}{g}_{\psi_2}\left(t_{t_2'};t'\right)\right|\le\int_{\gamma^2}\frac{\left|\Delta f\left(s_{t_1};t_{t_2'}\right)-\Delta f\left(s_{t_1'};t'\right)\right|}{|s_2-t_2'|}|d\psi(s_2)|=$$

$$=\int_{\gamma^2}\frac{\left|\Delta f\left(s_{t_1};t'\right)\right|}{|s_2-t_2'|}|d\psi(s_2)|\le\int_0^{l_2}\frac{\overset{1,1}{\omega}_f\left(|t_1-t_1'|,\breve\theta_2^{\psi_2}(\xi_2)\right)}{\breve\theta_2^{\psi_2}(\xi_2)}d\xi_2\le$$

$$\le\int_0^{\breve\theta_1^{\psi_1}(\varepsilon_1)}\int_0^{l_2}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1),\breve\theta_2^{\psi_2}(\xi_2)\right)}{\breve\theta_1^{\psi_1}(\xi_1)\cdot\breve\theta_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2. \qquad (22)$$

Applying the estimate for the difference (11) we get

$$\left|\Delta\overset{1,0}{g}_{\psi}\left(t_{t_2'};t'\right)\right|\le C\Bigg(\int_0^{\theta_1^{\psi_1}(\varepsilon_1)}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1)\right)}{\breve\theta_1^{\psi_1}(\xi_1)}d\xi_1+$$

$$+\varepsilon_1\int_{\theta_1^{\psi_1}(\varepsilon_1)}^{l_1}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1)\right)}{\left[\breve\theta_1^{\psi_1}(\xi_1)\right]^2}d\xi_1\Bigg). \qquad (23)$$

Taking the estimates (21)-(23) into account, with $\varepsilon_i=|t_i-t_1'|<\delta_i$, i=1,2. We finally obtain

$$\overset{1,0}{\omega}_{\bar f_\psi}(\delta_1)\le C\Bigg[\int_0^{\theta_1^{\psi_1}(\delta_1)}\int_0^{l_2}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1),\breve\theta_2^{\psi_2}(\xi_2)\right)}{\breve\theta_1^{\psi_1}(\xi_1)\cdot\breve\theta_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2+$$

$$+\delta_1\int_{\theta_1^{\psi_1}(\delta_1)}^{l_1}\int_0^{l_2}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1),\breve\theta_2^{\psi_2}(\xi_2)\right)}{\left[\breve\theta_1^{\psi_1}(\xi_1)\right]^2\cdot\breve\theta_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2+\int_0^{\theta_1^{\psi_1}(\delta_1)}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1)\right)}{\breve\theta_1^{\psi_1}(\xi_1)}d\xi_1+$$

$$+\int_{\theta_1^{\psi_1}(\delta_1)}^{l_1}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1)\right)}{\left[\breve\theta_1^{\psi_1}(\xi_1)\right]^2}d\xi_1\Bigg]. \qquad (24)$$

In similar way one may estimate $\Delta\bar f_\psi\left(t_{t_1'};t'\right)=\bar f_\psi(t_1';t_2)-\bar f_\psi(t_1';t_2')$. We have

$$\overset{0,1}{\omega}_{\bar f_\psi}(\delta_2)\le C\Bigg[\int_0^{\theta_2^{\psi_2}(\delta_2)}\int_0^{l_1}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1),\breve\theta_2^{\psi_2}(\xi_2)\right)}{\breve\theta_1^{\psi_1}(\xi_1)\cdot\breve\theta_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2+$$

$$+\delta_2\int_{\theta_2^{\psi_2}(\delta_2)}^{l_2}\int_0^{l_1}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_1^{\psi_1}(\xi_1),\breve\theta_2^{\psi_2}(\xi_2)\right)}{\left[\breve\theta_2^{\psi_2}(\xi_2)\right]^2\cdot\breve\theta_1^{\psi_1}(\xi_1)}d\xi_1 d\xi_2+\int_0^{\theta_2^{\psi_2}(\delta_2)}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_2^{\psi_2}(\xi_2)\right)}{\breve\theta_2^{\psi_2}(\xi_2)}d\xi_2+$$

$$+\int_{\theta_2^{\psi_2}(\delta_2)}^{l_2}\frac{\overset{1,1}{\omega}_f\left(\breve\theta_2^{\psi_2}(\xi_2)\right)}{\left[\breve\theta_2^{\psi_2}(\xi_2)\right]^2}d\xi_2\Bigg]. \qquad (25)$$

The last inequalities (20), (24) and (25) prove the theorem.

Let $\omega\in\Phi_{T^2}$. We introduce the linear space

$$K_\omega=\Big\{f\in C_\Delta:\overset{1,1}{\omega}_f(\delta_1,\delta_2)=O(\omega(\delta_1,\delta_2)),$$

$$\overset{1,0}{\omega}_f(\delta_1)=O(\omega(\delta_1,d_2)),\ \overset{0,1}{\omega}_f(\delta_2)=O(\omega(d_1,\delta_2))\Big\}$$

and equip it with the norm

$$\|f\|_{K_\omega}=\max\left\{\overset{1,1}{C}_f,\overset{1,0}{C}_f,\overset{0,1}{C}_f,\|f\|_{C_\Delta}\right\},$$

Where

$$\overset{1,1}{C}_f=\sup_{\delta_1,\delta_2>0}\frac{\overset{1,1}{\omega}_f(\delta_1,\delta_2)}{\omega(\delta_1,\delta_2)},\quad \overset{1,}{C}_f=\sup_{\delta_1>0}\frac{\overset{1,0}{\omega}_f(\delta_1)}{\omega(\delta_1,d_2)},$$

$$\overset{0,1}{C}_f=\sup_{\delta_2>0}\frac{\overset{0,1}{\omega}_f(\delta_2)}{\omega(d_1,\delta_2)}.$$

With respect to this norm K_ω is Banach space. It's easy to see that $K_\omega=K_{\omega_1}$ when $\omega_1\sim\omega_2$ up to equivalence of norms. In the case $\omega(\delta_1,\delta_2)=\delta_1^\alpha\delta_2^\beta$, $0<\alpha,\beta\le1$ we denote this space by $K_{\alpha,\beta}$ this class being treated in [3],[6], [13], [16]- [18].

Theorem 3. Let $d\psi(t)=F(t)dt$, where $F(t)$ is limiting value of function analytic in $D^\pm$ and continuous up to the boundary and let

$$\omega\in\Im_0\Phi=\Bigg\{\omega\in\Phi_{T^2}:\int_0^d\frac{\omega(\xi)}{\xi}d\theta^F(\xi)<\infty,$$

$$\int_0^{d_1}\frac{\omega(\xi_1,d_2)}{\xi_1}d\theta_1^F(\xi_1)<\infty,\ \int_0^{d_2}\frac{\omega(d_1,\xi_2)}{\xi_2}d\theta_2^F(\xi_2)<\infty,\Bigg\}.$$

Then $f\in K_\omega\Rightarrow\bar f_F\in K_z\left(\omega,\delta,\theta^F\right)$.

Corollary. Let $\theta_k^F(\delta)\sim\delta\ (k=1,2)$. Then under the assumption of Theorem 3 on ω we have $f\in K_\omega \Rightarrow \bar{f}_F\in K_z(\omega,\delta)$.

Finally let $\theta_k^F(\delta)\sim\delta\ (k=1,2)$,

$$\omega\in\{\omega\in\Im_0\Phi : Z(\omega;\delta_1,\delta_2)=O(\omega(\delta_1,\delta_2))\} \qquad (26)$$

Then the following theorem is valid.

Theorem 4. Let $f\in K_\omega$. Then under the assumptions (26), $\bar{f}\in K_\omega$ as well and $\|\bar{f}\|_{K_\omega}\le\|f\|_{K_\omega}$.

The proof of the theorem follows from Theorem 2, Corollary to Theorem 3 and the estimate

$$\|\bar{f}\|_{K_\omega}\le C\|f\|_{K_\omega}\left(\int_0^l \frac{\omega(\xi)}{\xi}d\xi+\int_0^{l_1}\frac{\omega(\xi_1,d_2)}{\xi_1}d\xi_1+\right.$$

$$\left.+\int_0^{l_2}\frac{\omega(d_1,\xi_2)}{\xi_2}d\xi_2\right)=C_1\|\bar{f}\|_{K_\omega}.$$

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